

A NOTE ON THE RELATIONSHIP BETWEEN THE SZLENK AND w^* -DENTABILITY INDICES OF ARBITRARY w^* -COMPACT SETS

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ABSTRACT. We prove the optimal estimate between the Szlenk and w^* -dentability indices of an arbitrary w^* -compact subset of the dual of a Banach space. For a given w^* -compact, convex subset K of the dual of a Banach space, we introduce a two player game the winning strategies of which determine the Szlenk index of K . We give applications to the w^* -dentability index of a Banach space and of an operator.

1. INTRODUCTION

Since its inception in [17], the Szlenk index has been an important tool in renorming theory [8], [16], [9]. In [7], the notion of ξ -asymptotically uniformly smooth operators was given, with the 0-asymptotically uniformly smooth notion generalizing the notion of an asymptotically uniformly smooth Banach space. It was shown in [7] that an operator $A : X \rightarrow Y$ has Szlenk index not exceeding $\omega^{\xi+1}$ if and only if there exists an equivalent norm $|\cdot|$ on Y making $A : X \rightarrow (Y, |\cdot|)$ ξ -asymptotically uniformly smooth. Applying this to the identity of a Banach space, we deduce that a Banach space X has Szlenk index not exceeding $\omega^{\xi+1}$ if and only if there exists an equivalent norm $|\cdot|$ on X such that $(X, |\cdot|)$ is ξ -asymptotically uniformly smooth.

Another index has been used to study the class of Asplund spaces, the w^* -dentability index. The w^* -dentability index is distinct from the Szlenk index, but each characterizes w^* -fragmentability of a w^* -compact set. Since both indices characterize w^* -fragmentability, it is natural to ask what relationship must exist between the indices. It follows immediately from the definitions that the Szlenk index of a set cannot exceed its w^* -dentability index. We discuss in the next section the different results obtained in the literature regarding the relationship between the w^* -dentability and Szlenk indices.

In what follows, $Sz(K)$ (resp. $Dz(K)$) will denote the Szlenk (resp. w^* -dentability index) of the set K .

Theorem 1. *Let X be a Banach space, let $K \subset X^*$ be w^* -compact, and let ξ be an ordinal.*

- (i) *If $Sz(K) \leq \omega^\xi$, then $Dz(K) \leq \omega^{1+\xi}$.*
- (ii) *Suppose that K is convex. Then $Dz(K) \leq \omega Sz(K)$, and if $Sz(K) \geq \omega^\omega$, $Dz(K) = Sz(K)$.*

As was discussed in [11], for every $n \in \mathbb{N} \cup \{0\}$, there exist Banach spaces X_n, Y_n such that $Sz(X_n) = Sz(Y_n) = \omega^n$ while $Dz(X_n) = \omega^{n+1}$ and $Dz(Y_n) = \omega^n$. These examples show the sharpness of Theorem 1.

In [1], it was shown that one can compute the Szlenk index of a separable Banach space containing no isomorph of ℓ_1 by considering convex combinations of the branches of trees of vectors satisfying a certain weak nullity condition. We also recall a particular two-player game played on a Banach space. For $\varepsilon > 0$ and every $n \in \mathbb{N}$, Player I chooses a subspace Z_1^n of X such that $\dim(X/Z_1^n) < \infty$, Player II chooses a vector $x_1^n \in B_{Z_1^n}$, ..., Player I chooses a subspace Z_n^n of X such that $\dim(X/Z_n^n) < \infty$, and Player II chooses a vector $x_n^n \in B_{Z_n^n}$. We say that Player II wins the game if for every $n \in \mathbb{N}$, $\|n^{-1} \sum_{i=1}^n x_i^n\| \geq \varepsilon$, and Player I wins otherwise. Then if X is a separable Banach space not containing ℓ_1 , the results of [1] combined with the results of [8] imply that $Sz(X) \leq \omega$ if and only if for every $\varepsilon > 0$, Player I has a winning strategy in this game. Since this game is determined, $Sz(X) > \omega$ if and only if for some $\varepsilon > 0$, Player II has a winning strategy in this game. Note that we require a certain “smallness” condition on a *specific* convex combination $n^{-1} \sum_{i=1}^n x_i^n$ of $(x_i^n)_{i=1}^n$.

In [4], the results of [1] were extended to allow one to compute the Szlenk index of an arbitrary w^* -compact subset of the dual of an arbitrary Banach space. In analogy to the game defined above, we wish to define for a given ordinal ξ a certain game the winning strategies of which determine whether the Szlenk index of an arbitrary w^* -compact set exceeds ω^ξ . Given a Banach space X , let \mathcal{D} denote the subspaces of X having finite codimension in X , and let \mathcal{K} denote the norm-compact subsets of X . Let $K \subset X^*$ be w^* -compact. Suppose that Λ is a set, T is a non-empty collection of non-empty sequences in Λ such that there does not exist an infinite sequence $(\zeta_i)_{i=1}^\infty \subset \Lambda$ all the finite initial segments of which lie in T (such a collection T is called a *non-empty, well-founded B-tree*). Assume also that $\mathbb{P} : T \rightarrow \mathbb{R}$ is a fixed function. For $\varepsilon > 0$, we let Player I choose $Z_1 \in \mathcal{D}$ and $\zeta_1 \in \Lambda$ such that $(\zeta_1) \in T$. Player II then chooses $C_1 \in \mathcal{K}$. Next, assuming $(\zeta_i)_{i=1}^n \in T$, $Z_1, \dots, Z_n \in \mathcal{D}$, and $C_1, \dots, C_n \in \mathcal{K}$ have been chosen, if $(\zeta_i)_{i=1}^n$ has no proper extensions in T , the game terminates. Otherwise Player I chooses $\zeta_{n+1} \in \Lambda$ such that $(\zeta_i)_{i=1}^{n+1} \in T$ and $Z_{n+1} \in \mathcal{D}$. Player II chooses $C_{n+1} \in \mathcal{K}$. Our assumptions on T yield that this game must terminate after finitely many turns. Let us assume the game terminates with the choices $(\zeta_i)_{i=1}^n, (Z_i)_{i=1}^n, (C_i)_{i=1}^n$. We say that Player II wins the game if there exist a sequence $(x_i)_{i=1}^n \in \prod_{i=1}^n (B_X \cap Z_i \cap C_i)$ and $x^* \in K$ such that

$$\operatorname{Re} x^* \left(\sum_{i=1}^n \mathbb{P}((\zeta_j)_{j=1}^i) x_i \right) \geq \varepsilon,$$

and let us say Player I wins otherwise. Let us refer to this as the $(\varepsilon, K, \mathbb{P})$ game on $T, \mathcal{D}, \mathcal{K}$. Our main result in this direction is the following.

Theorem 2. *For every ordinal ξ , there exists a non-empty, well-founded B-tree Γ_ξ on $[0, \omega^\xi]$ and a function $\mathbb{P}_\xi : \Gamma_\xi \rightarrow \mathbb{R}$ such that for any Banach space X and any w^* -compact $K \subset X^*$, $Sz(K) > \omega^\xi$ if and only if there exists $\varepsilon > 0$ such that Player II has a winning strategy in*

the $(\varepsilon, K, \mathbb{P}_\xi)$ -game on $\Gamma_\xi \mathcal{DK}$, and $Sz(K) \leq \omega^\xi$ if and only if for every $\varepsilon > 0$, Player I has a winning strategy in the $(\varepsilon, K, \mathbb{P}_\xi)$ game on $\Gamma_\xi \mathcal{DK}$.

2. DEFINITIONS

2.1. Definition of the indices. Let X be a Banach space and let $K \subset X^*$. For $\varepsilon > 0$, we let $s_\varepsilon(K)$ denote those $x^* \in K$ such that for every w^* -neighborhood V of x^* , $\text{diam}(V \cap K) > \varepsilon$. We let $d_\varepsilon(K)$ denote those $x^* \in K$ such that for every w^* -open slice S containing x^* , $\text{diam}(S \cap K) > \varepsilon$. Recall that a w^* -open slice is a subset of X^* of the form $\{y^* : \text{Re } y^*(x) > a\}$ for some $x \in X$ and $a \in \mathbb{R}$. We then define $s_\varepsilon^0(K) = K$, $s_\varepsilon^{\xi+1}(K) = s_\varepsilon(s_\varepsilon^\xi(K))$, and $s_\varepsilon^\xi(K) = \bigcap_{\zeta < \xi} s_\varepsilon^\zeta(K)$ when ξ is a limit ordinal. We set $Sz(K, \varepsilon) = \min\{\xi : s_\varepsilon^\xi(K) = \emptyset\}$ if this class of ordinals is non-empty, and we set $Sz(K, \varepsilon) = \infty$ otherwise. We let $Sz(K) = \sup_{\varepsilon > 0} Sz(K, \varepsilon)$, where we agree that $\xi < \infty$ for all ordinals ξ . If X is a Banach space, we let $Sz(X, \varepsilon) = Sz(B_{X^*}, \varepsilon)$ and $Sz(X) = Sz(B_{X^*})$. If $A : X \rightarrow Y$ is an operator, we let $Sz(A, \varepsilon) = Sz(A^*B_{Y^*}, \varepsilon)$ and $Sz(A) = Sz(A^*B_{Y^*})$. We define $d_\varepsilon^\xi(K)$, $Dz(K, \varepsilon)$, $Dz(K)$, etc., similarly. It is quite clear that $Sz(K) \leq Dz(K)$.

We recall that K is said to be w^* -fragmentable provided that for every non-empty subset L of K and every $\varepsilon > 0$, there exists a w^* -open subset U of X^* such that $U \cap L \neq \emptyset$ and $\text{diam}(U \cap L) < \varepsilon$. We say that K is w^* -dentable if for any non-empty subset L of K and every $\varepsilon > 0$, there exists a w^* -open slice S of X^* such that $S \cap L \neq \emptyset$ and $\text{diam}(S \cap L) < \varepsilon$. It is clear that K is w^* -fragmentable (resp. w^* -dentable) if and only if $Sz(K)$ (resp. $Dz(K)$) is an ordinal. Moreover, w^* -fragmentability and w^* -dentability are equivalent, which is a consequence of Theorem 1. Since these properties are equivalent, it is natural to consider the relationship between $Sz(K)$ and $Dz(K)$. Lancien [12] proved using descriptive set theoretic techniques that there exists a function $\phi : [0, \omega_1) \rightarrow [0, \omega_1)$ such that if $\xi < \omega_1$ and if X is a Banach space with $Sz(X) \leq \xi$, $Dz(X) \leq \phi(\xi)$. Raja [16] proved that for any Banach space (without assumption of countability of $Sz(X)$) that $Dz(X) \leq \omega^{Sz(X)}$. Hájek and Schlumprecht [11] showed that if $Sz(X)$ is countable, $Dz(X) \leq \omega Sz(X)$. The content of Theorem 1 extends this result of Hájek and Schlumprecht to the general case of an arbitrary w^* -compact, convex set K as opposed to the case $K = B_{X^*}$, and removes the hypothesis of countability of $Sz(K)$.

We note that the most interesting case, of course, is the case $K = B_{X^*}$. However, the case $K = A^*B_{Y^*}$ for an operator $A : X \rightarrow Y$ is also of interest. We refer the reader to [2], [7], and [6] for results concerning the Szlenk index of an operator, including renorming theorems for asymptotically uniformly smooth operators. However, to our knowledge, the w^* -dentability index of an operator has not been investigated.

2.2. B -trees. Given a set Λ , we let $\Lambda^{<\mathbb{N}}$ denote the finite sequences in Λ , including the empty sequence, \emptyset . We write $s \preceq t$ if s is an initial segment of t . If $t \in \Lambda^{<\mathbb{N}}$, we let $|t|$ denote the length of t and for $0 \leq i \leq |t|$, $t|_i$ is the initial segment of t having length i . If $\emptyset \neq t$, we let $t^- = t|_{|t|-1}$. We let $s \frown t$ denote the concatenation of s and t . A subset T of

$\Lambda^{<\mathbb{N}}$ is called a *tree* if for all $t \in T$ and $s \preceq t$, $s \in T$. A subset T of $\Lambda^{<\mathbb{N}} \setminus \{\emptyset\}$ will be called a *B-tree* provided that for any $t \in T$ and any $\emptyset \prec s \preceq t$, $s \in T$. We let $MAX(T)$ denote the members of T which are \prec -maximal and $T' = T \setminus MAX(T)$. We define $T^0 = T$, $T^{\xi+1} = (T^\xi)'$, and $T^\xi = \bigcap_{\zeta < \xi} T^\zeta$ when ξ is a limit ordinal. We say T is *well-founded* if there exists an ordinal ξ such that $T^\xi = \emptyset$, and we let $o(T)$ denote the smallest such ξ . If no such ξ exists, we say T is *ill-founded* and write $o(T) = \infty$. Note that $o(T) = \infty$ if and only if there exists an infinite sequence $(\zeta_i)_{i=1}^\infty \subset \Lambda$ such that $(\zeta_i)_{i=1}^n \in T$ for all $n \in \mathbb{N}$.

Recall that for any B -trees S, T , a function $\theta : S \rightarrow T$ is called *monotone* provided that for any $\emptyset \prec s \prec s_1 \in S$, $\theta(s) \prec \theta(s_1)$.

Given non-empty sets $\Lambda_1, \dots, \Lambda_k$, we identify the set $(\prod_{i=1}^k \Lambda_i)^{<\mathbb{N}}$ with the set $\{(t_i)_{i=1}^k \in \prod_{i=1}^k \Lambda_i^{<\mathbb{N}} : |t_1| = \dots = |t_k|\}$. The identification is obtained by identifying \emptyset with $(\emptyset, \dots, \emptyset)$ and, for $n > 0$,

$$((a_{1i}, \dots, a_{ki}))_{i=1}^n \leftrightarrow ((a_{1i})_{i=1}^n, (a_{2i})_{i=1}^n, \dots, (a_{ki})_{i=1}^n).$$

Let X be a Banach space and let T be a B -tree. Let us say that a collection $(x_t)_{t \in T} \subset X$ is *weakly null* provided that for every ordinal ξ , every $t \in (T \cup \{\emptyset\})^{\xi+1}$, and every $Z \leq X$ with $\dim(X/Z) < \infty$, there exists $s \in T^\xi$ with $s^- = t$ such that $x_s \in Z$.

We last define some B -trees which will be important for us. If $(\zeta_i)_{i=1}^n$ is a sequence of ordinals and ζ is an ordinal, we let $\zeta + (\zeta_i)_{i=1}^n = (\zeta + \zeta_i)_{i=1}^n$. If G is a collection of non-empty sequences of ordinals and ζ is an ordinal, we let $\zeta + G = \{\zeta + t : t \in G\}$. We let

$$\mathcal{T}_0 = \emptyset,$$

$$\mathcal{T}_{\xi+1} = \{(\xi + 1)^\wedge t : t \in \{\emptyset\} \cup \mathcal{T}_\xi\},$$

and if ξ is a limit ordinal, we let

$$\mathcal{T}_\xi = \bigcup_{\zeta < \xi} \mathcal{T}_{\zeta+1}.$$

Note that this union is a totally incomparable union. For each ordinal ξ , \mathcal{T}_ξ is a B -tree on $[0, \xi]$ with $o(\mathcal{T}_\xi) = \xi$.

Next, let

$$\Gamma_0 = \{(1)\},$$

$$\Gamma_{\xi+1} = \left\{ (\omega^\xi(n-1) + t_1)^\wedge \dots^\wedge (\omega^\xi(n-m) + t_m) : n \in \mathbb{N}, 1 \leq m \leq n, t_i \in \Gamma_\xi, \right. \\ \left. t_i \in MAX(\Gamma_\xi) \text{ for each } 1 \leq i < m \right\},$$

and when ξ is a limit ordinal,

$$\Gamma_\xi = \bigcup_{\zeta < \xi} (\omega^\zeta + \Gamma_{\zeta+1}).$$

For each ordinal ξ , Γ_ξ is a B -tree on $[1, \omega^\xi]$ with $o(\Gamma_\xi) = \omega^\xi$. We define $\mathbb{P}_\xi : \Gamma_\xi \rightarrow [0, 1]$ by letting $\mathbb{P}_0((1)) = 1$,

$$\mathbb{P}_{\xi+1}((\omega^\xi(n-1) + t_1)^\wedge \dots^\wedge (\omega^\xi(n-m) + t_m)) = \mathbb{P}_\xi(t_m)/n,$$

and

$$\mathbb{P}_\xi(\omega^\zeta + t) = \mathbb{P}_{\zeta+1}(t), \quad t \in \Gamma_{\zeta+1}.$$

We refer the reader to [5] for a discussion that these functions are well-defined and for every ordinal ξ and every $t \in \text{MAX}(\Gamma_\xi)$, $\sum_{s \preceq t} \mathbb{P}_\xi(s) = 1$.

3. GAMES ON WELL-FOUNDED B -TREES

Given a non-empty, well-founded B -tree T on the set Λ , let $R_T = \{\zeta \in \Lambda : (\zeta) \in T\}$. Given a non-empty, well-founded B -tree T and two non-empty sets \mathcal{D}, \mathcal{K} , we let $T.\mathcal{D}.\mathcal{K}$ denote the sequences $(\zeta_i, Z_i, C_i)_{i=1}^n$ such that $Z_i \in \mathcal{D}$, $C_i \in \mathcal{K}$, $(\zeta_i)_{i=1}^n \in T$. Let $T.\mathcal{D} = \{(\zeta_i, Z_i)_{i=1}^n : Z_i \in \mathcal{D}, (\zeta_i)_{i=1}^n \in T\}$. Note that $T.\mathcal{D}.\mathcal{K}$ and $T.\mathcal{D}$ are non-empty, well-founded B -trees with the same order as T . Given a subset $\mathcal{E} \subset \text{MAX}(T.\mathcal{D}.\mathcal{K})$, we define the \mathcal{E} -game on $T.\mathcal{D}.\mathcal{K}$ as follows: Player I chooses $Z_1 \in \mathcal{D}$ and $\zeta_1 \in R_T$. Player II chooses $C_1 \in \mathcal{K}$. Next, assuming that $Z_1, \dots, Z_n \in \mathcal{D}$, $C_1, \dots, C_n \in \mathcal{K}$, and $\zeta_1, \dots, \zeta_n \in \Lambda$ have been chosen such that $(\zeta_i)_{i=1}^n \in T$, if $(\zeta_i)_{i=1}^n \in \text{MAX}(T)$, the game terminates. Otherwise Player I chooses $Z_{n+1} \in \mathcal{D}$ and $\zeta_{n+1} \in \Lambda$ such that $(\zeta_i)_{i=1}^{n+1} \in T$ and player II chooses $C_{n+1} \in \mathcal{K}$. Since T is well-founded, the game terminates after some finite number of steps. Suppose that the game terminates after the choices $C_1, \dots, C_n \in \mathcal{K}$, $Z_1, \dots, Z_n \in \mathcal{D}$, and $\zeta_1, \dots, \zeta_n \in \Lambda$. Then Player I wins provided $(\zeta_i, Z_i, C_i)_{i=1}^n \in \text{MAX}(T.\mathcal{D}.\mathcal{K}) \setminus \mathcal{E}$, and Player II wins if $(\zeta_i, Z_i, C_i)_{i=1}^n \in \mathcal{E}$. We call such a game *on a non-empty, well-founded B -tree*.

A *strategy for Player I* is a function $\varphi : T'.\mathcal{D}.\mathcal{K} \cup \{\emptyset\} \rightarrow \Lambda \times \mathcal{D}$ such that if $\varphi(\emptyset) = (\zeta, Z)$, $\zeta \in R_T$, and if $\varphi((\zeta_i, Z_i, C_i)_{i=1}^n) = (\zeta_{n+1}, Z_{n+1})$, $(\zeta_i)_{i=1}^{n+1} \in T$. A sequence $(\zeta_i, Z_i, C_i)_{i=1}^n \in \text{MAX}(T.\mathcal{D}.\mathcal{K})$ is φ -*admissible* if $(\zeta_j, Z_j) = \varphi((\zeta_i, Z_i, C_i)_{i=1}^{j-1})$ for each $1 \leq j \leq n$. A strategy for Player I φ is called a *winning strategy for the \mathcal{E} game on $T.\mathcal{D}.\mathcal{K}$* provided that every φ -admissible sequence lies in $\text{MAX}(T.\mathcal{D}.\mathcal{K}) \setminus \mathcal{E}$. A *winning substrategy for Player I for the \mathcal{E} game on $T.\mathcal{D}.\mathcal{K}$* is a subset S of $T'.\mathcal{D}.\mathcal{K} \cup \{\emptyset\}$ containing \emptyset and a function $\phi : S \rightarrow \Lambda \times \mathcal{D}$ such that, if $(\zeta, Z) = \phi(\emptyset)$,

- (i) $S = \{\emptyset\} \cup \{t \in T'.\mathcal{D}.\mathcal{K} : (\exists C \in \mathcal{K})((\zeta, Z, C) \preceq t)\}$,
- (ii) $\zeta \in R_T$,
- (iii) if $t = (\zeta_i, Z_i, C_i)_{i=1}^n \in S$ and $(\zeta_{n+1}, Z_{n+1}) = \phi(t)$, then $(\zeta_i)_{i=1}^{n+1} \in T$,
- (iv) if $(\zeta_i, Z_i, C_i)_{i=1}^n \in \text{MAX}(T.\mathcal{D}.\mathcal{K})$, $(\zeta_1, Z_1) = (\zeta, Z)$, and $(\zeta_j, Z_j) = \phi((\zeta_i, Z_i, C_i)_{i=1}^{j-1})$ for each $1 \leq j \leq n$, then $t \notin \mathcal{E}$.

Note that if Player I has a winning substrategy for the \mathcal{E} game on $T.\mathcal{D}.\mathcal{K}$, then Player I has a winning strategy. Indeed, given a winning substrategy $\phi : S \rightarrow \Lambda \times \mathcal{D}$, we fix any $Z' \in \mathcal{D}$ and define a strategy $\varphi : T'.\mathcal{D}.\mathcal{K} \rightarrow \Lambda \times \mathcal{D}$ by letting $\varphi|_S = \phi$ and, if $t = (\zeta_i, Z_i, C_i)_{i=1}^n \in T'.\mathcal{D}.\mathcal{K} \setminus S$, we let $\varphi(t) = (\zeta_{n+1}, Z')$ for any $\zeta_{n+1} \in \Lambda$ such that $(\zeta_i)_{i=1}^{n+1} \in T$. Such a ζ_{n+1} exists since $(\zeta_i)_{i=1}^n \in T'$. Let $(\zeta, Z) = \phi(\emptyset)$. It is straightforward to verify that this is a strategy for Player I. Since any φ -admissible sequence $(\zeta_i, Z_i, C_i)_{i=1}^n$ satisfies $(\zeta_1, Z_1) = (\zeta, Z)$, property (iv) of winning substrategy guarantees that $(\zeta_i, Z_i, C_i)_{i=1}^n \in \text{MAX}(T.\mathcal{D}.\mathcal{K}) \setminus \mathcal{E}$.

A *strategy for Player II* is a \mathcal{K} -valued function ψ on the set of all pairs $(t, (\zeta_{n+1}, Z_{n+1}))$ such that $t \in \{\emptyset\} \cup T.\mathcal{D}.\mathcal{K}$, $(\zeta_{n+1}, Z_{n+1}) \in \Lambda \times \mathcal{D}$, and if $t = (\zeta_i, Z_i, C_i)_{i=1}^n$, $(\zeta_i)_{i=1}^{n+1} \in T$. A sequence $(\zeta_i, Z_i, C_i)_{i=1}^n \in \text{MAX}(T.\mathcal{D}.\mathcal{K})$ is ψ -*admissible* provided that for every $1 \leq k \leq n$, $\psi((\zeta_i, Z_i, C_i)_{i=1}^{k-1}, (\zeta_k, Z_k)) = C_k$. A strategy for player II ψ is called a *winning strategy for the \mathcal{E} game on $T.\mathcal{D}.\mathcal{K}$* provided that every ψ -admissible sequence lies in \mathcal{E} . Obviously for a given subset \mathcal{E} of $\text{MAX}(T.\mathcal{D}.\mathcal{K})$, Player I and Player II cannot both have a winning strategy.

Proposition 3.1. *Every game on a non-empty, well-founded B-tree is determined. That is, exactly one of Player I and Player II has a winning strategy.*

Proof. We prove by induction on $\xi \geq 1$ that if T is a non-empty, well-founded B -tree with $o(T) \leq \xi$, then for any $\mathcal{E} \subset \text{MAX}(T.\mathcal{D}.\mathcal{K})$, either Player I has a winning strategy or Player II has a winning strategy. Assume that for some ordinal ξ and every $1 \leq \gamma < \xi$, the statement is true hypothesis is true for γ . Let T be a non-empty, well-founded B -tree with $o(T) = \xi$. For every $\zeta \in R_T$, let $T(\zeta)$ denote those non-empty sequences t such that $(\zeta)^\frown t \in T$. Note that $T(\zeta)$ is a B -tree with $o(T(\zeta)) < \xi$, and $T(\zeta) = \emptyset$ if and only if $\zeta \in \text{MAX}(T)$. Given $\zeta \in R_T$, $Z \in \mathcal{D}$, and $C \in \mathcal{K}$, let $\mathcal{E}(\zeta, Z, C)$ denote those non-empty sequences $(\zeta_i, Z_i, C_i)_{i=1}^n$ such that $(\zeta, Z, C)^\frown (\zeta_i, Z_i, C_i)_{i=1}^n \in \mathcal{E}$. Let W denote the set of those $(\zeta, Z) \in T.\mathcal{D}$ such that either

- (i) $\zeta \in \text{MAX}(T)$ and for every $C \in \mathcal{K}$, $(\zeta, Z, C) \in \text{MAX}(T.\mathcal{D}.\mathcal{K}) \setminus \mathcal{E}$, or
- (ii) $\zeta \notin \text{MAX}(T)$ and for every $C \in \mathcal{K}$, Player I has a winning strategy in the $\mathcal{E}(\zeta, Z, K)$ game on $T(\zeta).\mathcal{D}.\mathcal{K}$.

By the inductive hypothesis, if $(\zeta, Z) \in T.\mathcal{D} \setminus W$, then either

- (i) $\zeta \in \text{MAX}(T)$ and there exists $C \in \mathcal{K}$ such that $(\zeta, Z, C) \in \mathcal{E}$, or
- (ii) $\zeta \notin \text{MAX}(T)$ and there exists $C \in \mathcal{K}$ such that Player II has a winning strategy in the $\mathcal{E}(\zeta, Z, C)$ game on $T(\zeta).\mathcal{D}.\mathcal{K}$.

It is obvious that Player I has a winning strategy in the \mathcal{E} game on $T.\mathcal{D}.\mathcal{K}$ if $W \neq \emptyset$, and Player II has a winning strategy in the \mathcal{E} game on $T.\mathcal{D}.\mathcal{K}$ if $W = \emptyset$. For completeness, we define the strategies in each case.

Suppose $W \neq \emptyset$. Fix $(\zeta, Z) \in W$ and let $S = \{\emptyset\} \cup \{t \in T'.\mathcal{D}.\mathcal{K} : (\exists C \in \mathcal{K})((\zeta, Z, C) \preceq t)\}$. If $\zeta \in \text{MAX}(T)$, then we define $\phi(\emptyset) = (\zeta, Z)$. Next, suppose $\zeta \notin \text{MAX}(T)$. For each $C \in \mathcal{K}$, fix a winning strategy $\varphi_C : T(\zeta)'.\mathcal{D}.\mathcal{K} \rightarrow \Lambda \times \mathcal{D}$ in the $\mathcal{E}(\zeta, Z, K)$ game on $T(\zeta).\mathcal{D}.\mathcal{K}$. Let $\phi(\emptyset) = (\zeta, Z)$ and for each $C \in \mathcal{K}$ and each extension $s = (\zeta, Z, C)^\frown t \in T'.\mathcal{D}.\mathcal{K}$ of (ζ, Z, C) , let $\phi(s) = \phi_C(t)$. In either case, we have produced a winning substrategy, which we may extend to a winning strategy by the remarks preceding the proposition.

Next, suppose $W = \emptyset$. Fix $C' \in \mathcal{K}$. Fix $(\zeta, Z) \in R_T \times \mathcal{D}$. If $(\zeta) \in \text{MAX}(T)$, fix $C_{\zeta, Z} \in \mathcal{K}$ such that $(\zeta, Z, C_{\zeta, Z}) \in \mathcal{E}$ and let $\psi(\emptyset, (\zeta, Z)) = C_{\zeta, Z}$. If $(\zeta) \in T'$, let $\psi(\emptyset, (\zeta, Z)) = C_{\zeta, Z}$, where $C_{\zeta, Z} \in \mathcal{K}$ is such that Player II has a winning strategy in the $\mathcal{E}(\zeta, Z, C_{\zeta, Z})$ game on $T(\zeta).\mathcal{D}.\mathcal{K}$, and let $\psi_{\zeta, Z}$ be a winning strategy on the appropriate domain. For $s = (\zeta, Z, C)^\frown (\zeta_i, Z_i, C_i)_{i=1}^n$ and $(\zeta_{n+1}, Z_{n+1}) \in \Lambda \times \mathcal{D}$ such that $(\zeta, \zeta_1, \dots, \zeta_{n+1}) \in T$, let

$\psi(s, (\zeta_{n+1}, Z_{n+1})) = C'$ if $C \neq C_{\zeta, Z}$ and $\psi(s, (\zeta_{n+1}, Z_{n+1})) = \psi_{\zeta, Z}((\zeta_i, Z_i, C_i)_{i=1}^n, (\zeta_{n+1}, Z_{n+1}))$ if $C = C_{\zeta, Z}$. This defines a winning strategy for Player II. \square

4. SZLENK GAMES

In Sections 3, 4, and 5, X will be a fixed Banach space, \mathcal{D} will be the subspaces of X having finite codimension in X , and \mathcal{K} will denote set of norm compact subsets of X . Given a non-empty, well-founded B -tree T and a collection $(x_{(s,t)})_{(s,t) \in \Pi(T, \mathcal{D})} \subset X$, we say the collection is *normally weakly null* provided that for any $s = (\zeta_i, Z_i)_{i=1}^n \in T \cdot \mathcal{D}$ and any t such that $(s, t) \in \Pi(T, \mathcal{D})$, $x_{(s,t)} \in B_{Z_n}$. We will also use normally weakly null to describe a collection $(x_s)_{s \in T \cdot \mathcal{D}}$ such that if $s = (\zeta_i, Z_i)_{i=1}^n \in T \cdot \mathcal{D}$, $x_s \in Z_i$. This is a special case of the previous definition in which the collection $(x_{(s,t)})_{(s,t) \in \Pi(T, \mathcal{D})}$ is such that $x_{(s,t)}$ is independent of t .

4.1. Determination of Szlenk index by games. Given $K \subset X^*$, $\varepsilon \in \mathbb{R}$, a B -tree T , and a function $\mathbb{P} : T \rightarrow \mathbb{R}$, we let $\mathcal{E}_{K, \varepsilon}(T \cdot \mathcal{D} \cdot \mathcal{K}, \mathbb{P})$ denote those $(\zeta_i, Z_i, C_i)_{i=1}^n \in \text{MAX}(T \cdot \mathcal{D} \cdot \mathcal{K})$ such that there exist $x^* \in K$ and $(x_i)_{i=1}^n \in \prod_{i=1}^n (B_X \cap Z_i \cap C_i)$ such that

$$\text{Re } x^* \left(\sum_{i=1}^n \mathbb{P}((\zeta_j)_{j=1}^i) x_i \right) \geq \varepsilon.$$

Given a function $\mathbb{P} : T \rightarrow \mathbb{R}$, we will consider the function \mathbb{P} to be also defined on $T \cdot \mathcal{D}$ by $\mathbb{P}((\zeta_i, Z_i)_{i=1}^n) = \mathbb{P}((\zeta_i)_{i=1}^n)$.

Lemma 4.1. *Fix a non-empty, well-founded B -tree T , a function $\mathbb{P} : T \rightarrow \mathbb{R}$, $\varepsilon \in \mathbb{R}$, and a subset K of X^* . If Player II has a winning strategy in the $\mathcal{E}_{K, \varepsilon}(T \cdot \mathcal{D} \cdot \mathcal{K}, \mathbb{P})$ game, then there exist a normally weakly null collection $(x_{(s,t)})_{(s,t) \in \Pi(T, \mathcal{D})} \subset B_X$, a collection $(x_t^*)_{t \in \text{MAX}(T \cdot \mathcal{D})} \subset K$, and a collection $(C_s)_{s \in T \cdot \mathcal{D}} \subset \mathcal{K}$ such that*

(i) *for every $t \in \text{MAX}(T \cdot \mathcal{D})$,*

$$\text{Re } x_t^* \left(\sum_{s \preceq t} \mathbb{P}(s) x_{(s,t)} \right) \geq \varepsilon,$$

and

(ii) *for every $s \in T \cdot \mathcal{D}$ and any maximal extension $t \in T \cdot \mathcal{D}$ of s , $x_{(s,t)} \in C_s$.*

Proof. Fix a winning strategy ψ for Player II in the $\mathcal{E}_{K, \varepsilon}(T \cdot \mathcal{D} \cdot \mathcal{K}, \mathbb{P})$ game. We first define $C_s \in \mathcal{K}$ for $s \in T \cdot \mathcal{D}$ by induction on $|s|$. If $|s| = 1$, write $s = (\zeta, Z)$ and let $C_s = \psi(\emptyset, (\zeta, Z))$. Next, suppose that for some $j \in \mathbb{N}$ and some sequence $s = (\zeta_i, Z_i)_{i=1}^{j+1} \in T \cdot \mathcal{D}$, $C_{s|_i}$ has been defined for each $1 \leq i \leq j$. Let $C_s = \psi((\zeta_i, Z_i, C_{s|_i})_{i=1}^j, (\zeta_{j+1}, Z_{j+1}))$. This completes the definition of $(C_s)_{s \in T \cdot \mathcal{D}}$. Note that with this definition, for every $t = (\zeta_i, Z_i)_{i=1}^n \in \text{MAX}(T \cdot \mathcal{D})$, the sequence $(\zeta_i, Z_i, C_{t|_i})_{i=1}^n$ is ψ -admissible and therefore lies in $\mathcal{E}_{K, \varepsilon}(T \cdot \mathcal{D} \cdot \mathcal{K}, \mathbb{P})$. Thus there exists $x_t^* \in K$ and a sequence $(x_i^t)_{i=1}^{|t|} \in \prod_{i=1}^{|t|} (B_X \cap Z_i \cap C_{t|_i})$ such that

$$\text{Re } x_t^* \left(\sum_{s \preceq t} \mathbb{P}(s) x_{|s|}^t \right) \geq \varepsilon.$$

Letting $x_{(s,t)} = x_{|s|}^t$ finishes the proof. □

Given B -trees S, T , we say a pair of functions $\theta : S.\mathcal{D} \rightarrow T.\mathcal{D}$, $e : \text{MAX}(S) \rightarrow \text{MAX}(T)$ is an *extended pruning* provided it is monotone, if $s = (\zeta_i, Z_i)_{i=1}^m$ and $\theta(s) = (\mu_i, W_i)_{i=1}^n$, $W_n \subset Z_m$, and for any $(s, t) \in \Pi(S.\mathcal{D})$, $\theta(s) \preceq e(t)$. We will write $(\theta, e) : S.\mathcal{D} \rightarrow T.\mathcal{D}$ to denote an extended pruning.

Lemma 4.2. *For any ordinal $\gamma > 0$ and any finite subset P_1, \dots, P_n of $\text{MAX}(\mathcal{T}_\gamma.\mathcal{D})$, there exist an extended pruning $(\theta, e) : \mathcal{T}_\gamma.\mathcal{D} \rightarrow \mathcal{T}_\gamma.\mathcal{D}$ and $1 \leq i \leq n$ such that $e(\text{MAX}(\mathcal{T}_\gamma.\mathcal{D})) \subset P_i$.*

Proof. It was shown in [3] that for any $0 < \xi \leq \gamma$, there exists a function $\phi : \mathcal{T}_\xi \rightarrow \mathcal{T}_\gamma$ such that for any $\emptyset \prec s \preceq s_1 \in \mathcal{T}_\xi$, $\phi(s) \prec \phi(s_1)$. From this we easily deduce that for any $0 < \xi \leq \gamma$, there exists an extended pruning $(\theta, e) : \mathcal{T}_\xi.\mathcal{D} \rightarrow \mathcal{T}_\gamma.\mathcal{D}$. Indeed, we first note that the function $\varphi : \mathcal{T}_\xi \rightarrow \mathcal{T}_\gamma$ given by $\varphi(s) = \phi(s)|_s$ is well-defined and still has the property that for any $\emptyset \prec s \preceq s_1 \in \mathcal{T}_\xi$, $\varphi(s) \prec \varphi(s_1)$, and φ preserves lengths. We may then define $\theta((\zeta_i, Z_i)_{i=1}^n) = (\mu_i, Z_i)_{i=1}^n$, where $\varphi((\zeta_i)_{i=1}^n) = (\mu_i)_{i=1}^n$. Then for every $t \in \text{MAX}(\mathcal{T}_\xi)$, let $e(t)$ be any maximal extension of $\theta(t)$, at least one of which exists by well-foundedness.

Recall that $\mathcal{T}_1.\mathcal{D} = \{(1, Z) : Z \in \mathcal{D}\}$. There exists $1 \leq i \leq n$ such that the set $M = \{Z : (1, Z) \in P_i\}$ is cofinal in \mathcal{D} . This means that for any $Z \in \mathcal{D}$, there exists $W_Z \in M$ such that $W_Z \leq Z$ and we may let $\theta((1, Z)) = e((1, Z)) = (1, W_Z)$. Then $e(\text{MAX}(\mathcal{T}_1.\mathcal{D})) \subset P_i$.

Next, suppose γ is a limit ordinal and the result holds for all $\xi < \gamma$. Recall that $\mathcal{T}_\gamma.\mathcal{D} = \cup_{\xi < \gamma} \mathcal{T}_{\xi+1}.\mathcal{D}$, and this is a disjoint union. For every $\xi < \gamma$, there exist an extended pruning $(\theta_\xi, e_\xi) : \mathcal{T}_{\xi+1}.\mathcal{D} \rightarrow \mathcal{T}_{\xi+1}.\mathcal{D}$ and $1 \leq i_\xi \leq n$ such that $e_\xi(\text{MAX}(\mathcal{T}_{\xi+1}.\mathcal{D})) \subset P_{i_\xi}$. There exists $1 \leq i \leq n$ such that $M = \{\xi < \gamma : i_\xi = i\}$ has supremum γ . For every $\xi < \gamma$, fix $\eta_\xi \in M$ with $\xi < \eta_\xi$ and an extended pruning $(\theta'_{\xi}, e'_{\xi}) : \mathcal{T}_{\eta_\xi+1}.\mathcal{D} \rightarrow \mathcal{T}_{\eta_\xi+1}.\mathcal{D}$, as we may by the first paragraph of the proof. Let $\theta|_{\mathcal{T}_{\xi+1}.\mathcal{D}} = \theta_{\eta_\xi+1} \circ \theta'_\xi$ and $e|_{\text{MAX}(\mathcal{T}_{\xi+1}.\mathcal{D})} = e_{\eta_\xi+1} \circ e'_\xi$. Then $e(\text{MAX}(\mathcal{T}_\gamma.\mathcal{D})) \subset P_i$.

Next, assume the result holds for an ordinal $\xi > 0$ and $\gamma = \xi + 1$. For $Z \in \mathcal{D}$, identifying $\{(\gamma, Z)^\wedge t : t \in \mathcal{T}_\xi.\mathcal{D}\}$ with $\mathcal{T}_\xi.\mathcal{D}$, we may find an extended pruning $(\theta_Z, e_Z) : \mathcal{T}_\xi.\mathcal{D} \rightarrow \mathcal{T}_\xi.\mathcal{D}$ and $1 \leq i_Z \leq n$ such that $\{(\gamma, Z)^\wedge e_Z(t) : t \in \text{MAX}(\mathcal{T}_\xi.\mathcal{D})\} \subset P_{i_Z}$. There exists $1 \leq i \leq n$ such that $M = \{Z \in \mathcal{D} : i_Z = i\}$ is cofinal in \mathcal{D} . For $Z \in \mathcal{D}$, fix $W_Z \in M$ such that $W_Z \leq Z$ and define

$$\theta((\gamma, Z)) = (\gamma, W_Z),$$

$$\theta((\gamma, Z)^\wedge t) = (\gamma, W_Z)^\wedge \theta_{W_Z}(t), \quad t \in \mathcal{T}_\xi.\mathcal{D},$$

$$e((\gamma, Z)^\wedge t) = (\gamma, W_Z)^\wedge e_{W_Z}(t), \quad t \in \text{MAX}(\mathcal{T}_\xi.\mathcal{D}).$$

This is an extended pruning with $e(\text{MAX}(\mathcal{T}_\gamma.\mathcal{D})) \subset P_i$. □

Lemma 4.3. *Fix an ordinal $\xi > 0$. Suppose that T is a well-founded, non-empty B -tree with $o(T) \geq \xi$ and $(x_{(s,t)})_{(s,t) \in \Pi(T, \mathcal{D})} \subset B_X$ is normally weakly null. Suppose also that for every $s \in T, \mathcal{D}$, C_s is a norm compact subset of X such that for every maximal extension t of s , $x_{(s,t)} \in C_s$. Then for any $\delta > 0$, there exists a collection $(x'_t)_{t \in \mathcal{T}_\xi, \mathcal{D}} \subset B_X$ which is normally weakly null and an extended pruning $(\theta, e) : \mathcal{T}_\xi, \mathcal{D} \rightarrow T, \mathcal{D}$ such that for every $(s, t) \in \Pi(\mathcal{T}_\xi, \mathcal{D})$, $\|x'_s - x_{(\theta(s), e(t))}\| < \delta$.*

Proof. We induct on ξ . First suppose $\xi = 1$. Recall that $\mathcal{T}_1, \mathcal{D} = \{(1, Z) : Z \in \mathcal{D}\}$, so that $\Pi(\mathcal{T}_1, \mathcal{D}) = \{((1, Z), (1, Z)) : Z \in \mathcal{D}\}$. Fix any $\zeta \in R_T$, as we may, since $o(T) \geq 1$. For every $Z \in \mathcal{D}$, fix a maximal extension t_Z of (ζ, Z) . Let $\theta((1, Z)) = (\zeta, Z)$, $e((1, Z)) = t_Z$, and let $x'_{(1, Z)} = x_{\theta((1, Z)), e((1, Z))}$. The conclusions are easily seen to be satisfied in this case with $\delta = 0$.

The limit ordinal case is trivial, since $\mathcal{T}_\xi, \mathcal{D} = \cup_{\zeta < \xi} \mathcal{T}_{\zeta+1}, \mathcal{D}$ is an incomparable union.

Assume $\gamma > 0$, the statement holds for γ , and $\xi = \gamma + 1$. Fix any ζ such that $(\zeta) \in T^\gamma$. Let S denote those non-empty sequences u such that $(\zeta)^\frown s \in T$. Fix $Z \in \mathcal{D}$. Since $(\zeta, Z) \in T^\gamma$, $o(S, \mathcal{D}) \geq \gamma$ and $(x_{((\zeta, Z)^\frown s, (\zeta, Z)^\frown t)})_{(s,t) \in \Pi(S, \mathcal{D})}$ is normally weakly null. Applying the inductive hypothesis to this collection and the sets $(C_{((\zeta, Z)^\frown s, (\zeta, Z)^\frown t)})_{(s,t) \in \Pi(S, \mathcal{D})}$, we deduce the existence of a normally weakly null collection $(x^Z_{(s,t)})_{(s,t) \in \Pi(\mathcal{T}_\gamma, \mathcal{D})} \subset B_X$ and an extended pruning $(\theta_Z, e_Z) : \mathcal{T}_\gamma, \mathcal{D} \rightarrow S, \mathcal{D}$ such that for every $(s, t) \in \Pi(\mathcal{T}_\gamma, \mathcal{D})$,

$$\|x^Z_{(s,t)} - x_{((\zeta, Z)^\frown \theta_Z(s), (\zeta, Z)^\frown e_Z(t))}\| < \delta.$$

Next, let $(v_i)_{i=1}^n$ be a finite $\delta/2$ -net of $C_{(\zeta, Z)}$. Then if

$$P_i = \{t \in \text{MAX}(\mathcal{T}_\gamma, \mathcal{D}) : \|v_i - x_{((\zeta, Z)^\frown \theta_Z(t), (\zeta, Z)^\frown e_Z(t))}\| < \delta/2\},$$

by Lemma 4.2, there exists an extended pruning $(\theta'_Z, e'_Z) : \mathcal{T}_\gamma, \mathcal{D} \rightarrow \mathcal{T}_\gamma, \mathcal{D}$ and $1 \leq i_Z \leq n$ such that $e'_Z(\text{MAX}(\mathcal{T}_\gamma, \mathcal{D})) \subset P_{i_Z}$. Fix $t_0 \in \text{MAX}(\mathcal{T}_\gamma, \mathcal{D})$ and let

$$\begin{aligned} x'_{(\xi, Z)} &= x_{((\zeta, Z)^\frown \theta'_Z(t_0), (\zeta, Z)^\frown e'_Z(t_0))}, & x'_{((\xi, Z)^\frown \theta_Z \circ \theta'_Z(t))}, \\ \theta((\xi, Z)) &= (\zeta, Z), & \theta((\xi, Z)^\frown t) = (\zeta, Z)^\frown \theta_Z \circ \theta'_Z(t), \\ e((\xi, Z)^\frown t) &= e_Z \circ e'_Z(t). \end{aligned}$$

□

Remark 4.4. Let \mathcal{N} denote any weak neighborhood basis at 0 in X . Given a non-empty B -tree T , let us say that $(x_t)_{t \in T, \mathcal{N}} \subset B_X$ is *usually weakly null* if for every $t = (\zeta_i, U_i)_{i=1}^n \in T, \mathcal{N}$, $x_t \in U_n$. Note that for any $\delta > 0$, there exist functions $\rho : \mathcal{D} \rightarrow \mathcal{N}$ and $\varrho : \mathcal{N} \rightarrow \mathcal{D}$ such that for any $Z \in \mathcal{D}$ and $U \in \mathcal{N}$, $B_Z \subset \rho(Z) \cap B_X$ and for any $x \in U \cap B_X$, there exists $y \in B_{\varrho(U)}$ with $\|x - y\| < \delta$. For $\varepsilon > 0$ and $\emptyset \neq K \subset X^*$, let $\mathcal{H}_\varepsilon^K$ denote the empty sequence together with those sequences $(x_i)_{i=1}^n \in B_X^{< \mathbb{N}}$ such that there exists $x^* \in K$ such that for every $1 \leq i \leq n$, $\text{Re } x^*(x_i) \geq \varepsilon$. The main theorem of [4] is the existence of a constant $c > 0$ such that

- (i) if there exists a usually weakly null $(x_t)_{t \in \mathcal{T}_{\omega^\xi} \cdot \mathcal{N}} \subset B_X$ such that for every $t \in \mathcal{T}_{\omega^\xi} \cdot \mathcal{N}$, $(x_{t|i})_{i=1}^{|t|} \in \mathcal{H}_\varepsilon^K$, then $Sz(K, \varepsilon_1) > \omega^\xi$ for every $0 < \varepsilon_1 < \varepsilon$, and
- (ii) if $Sz(K, c\varepsilon) > \omega^\xi$, there exists a usually weakly null $(x_t)_{t \in \mathcal{T}_{\omega^\xi} \cdot \mathcal{N}} \subset B_X$ such that for every $t \in \mathcal{T}_{\omega^\xi} \cdot \mathcal{N}$, $(x_{t|i})_{i=1}^{|t|} \in \mathcal{H}_\varepsilon^K$.

This combined with the existence of the functions ρ, ϱ above, we deduce that

- (i) if there exists a normally weakly null $(x_t)_{t \in \mathcal{T}_{\omega^\xi} \cdot \mathcal{D}} \subset B_X$ such that for every $t \in \mathcal{T}_{\omega^\xi} \cdot \mathcal{D}$, $(x_{t|i})_{i=1}^{|t|} \in \mathcal{H}_\varepsilon^K$, then $Sz(K, \varepsilon_1) > \omega^\xi$ for every $0 < \varepsilon_1 < \varepsilon$, and
- (ii) for any $c' > c$, if $Sz(K, c'\varepsilon) > \omega^\xi$, then there exists a normally weakly null $(x_t)_{t \in \mathcal{T}_{\omega^\xi} \cdot \mathcal{D}} \subset B_X$ such that for every $t \in \mathcal{T}_{\omega^\xi} \cdot \mathcal{D}$, $(x_{t|i})_{i=1}^{|t|} \in \mathcal{H}_\varepsilon^K$.

From this, it follows that if $Sz(K) > \omega^\xi$, then there exists $\varepsilon > 0$ such that Player II has a winning strategy in the $\mathcal{E}_{K,\varepsilon}(\Gamma_\xi \cdot \mathcal{D} \cdot \mathcal{K}, \mathbb{P}_\xi)$ game. Indeed, there exists $\varepsilon > 0$ such that $Sz(K, 2c\varepsilon) > \omega^\xi$, and a normally weakly null $(x_t)_{t \in \mathcal{T}_{\omega^\xi} \cdot \mathcal{D}} \subset B_X$ such that for every $t \in \mathcal{T}_{\omega^\xi} \cdot \mathcal{D}$, $(x_{t|i})_{i=1}^{|t|} \in \mathcal{H}_\varepsilon^K$. Since there exists a length-preserving, monotone $\theta : \Gamma_\xi \rightarrow \mathcal{T}_{\omega^\xi}$, we may let $\phi : \Gamma_\xi \cdot \mathcal{D} \rightarrow \mathcal{T}_{\omega^\xi} \cdot \mathcal{D}$ be given by $\phi((\zeta_i, Z_i)_{i=1}^n) = (\mu_i, Z_i)_{i=1}^n$, where $(\mu_i)_{i=1}^n = \phi((\zeta_i)_{i=1}^n)$. By relabeling, we may assume we have a normally weakly null $(x_t)_{t \in \Gamma_\xi \cdot \mathcal{D}} \subset B_X$ such that for every $t \in \Gamma_\xi \cdot \mathcal{D}$, $(x_{t|i})_{i=1}^{|t|} \in \mathcal{H}_\varepsilon^K$. We define a winning strategy ψ for Player II in the $\mathcal{E}_{K,\varepsilon}(\Gamma_\xi \cdot \mathcal{D} \cdot \mathcal{K}, \mathbb{P}_\xi)$ game. Let $\psi(\emptyset, (\zeta, Z)) = \{x_{(\zeta, Z)}\}$ and $\psi((\zeta_i, Z_i, C_i)_{i=1}^n, (\zeta_{n+1}, Z_{n+1})) = \{x_{(\zeta_i, Z_i)_{i=1}^{n+1}}\}$ for any compact sets C_1, \dots, C_n . Fix $t = (\zeta_i, Z_i, C_i)_{i=1}^n \in \text{MAX}(\Gamma_\xi \cdot \mathcal{D} \cdot \mathcal{K})$ which is ψ -admissible, let $s = (\zeta_i, Z_i)_{i=1}^n$, and note that for each $1 \leq i \leq n$, $C_i = \{x_{s|i}\}$. Then $(x_{s|i})_{i=1}^n \in \prod_{i=1}^n (B_X \cap Z_i \cap C_i)$. Since $(x_{s|i})_{i=1}^n \in \mathcal{H}_\varepsilon^K$, there exists $x^* \in K$ such that for every $1 \leq i \leq n$, $\text{Re } x^*(x_i) \geq \varepsilon$, and

$$\text{Re } x^* \left(\sum_{s \preceq t} \mathbb{P}_\xi(s) x_s \right) \geq \varepsilon.$$

Thus $t \in \mathcal{E}_{K,\varepsilon}(\Gamma_\xi \cdot \mathcal{D} \cdot \mathcal{K}, \mathbb{P}_\xi)$.

The next corollary shows the converse of this fact.

Corollary 4.5. *Suppose that $K \subset X^*$ is w^* -compact, $\varepsilon > 0$, and ξ is an ordinal such that Player II has a winning strategy in the $\mathcal{E}_{K,\varepsilon}(\Gamma_\xi \cdot \mathcal{D} \cdot \mathcal{K}, \mathbb{P}_\xi)$ game. Then for any $0 < \varepsilon_1 < \varepsilon$, $Sz(K, \varepsilon_1) > \omega^\xi$.*

Proof. Fix $\varepsilon_1 < \varepsilon' < \varepsilon$. By Lemma 4.1, we may fix a normally weakly null $(x_{(s,t)})_{(s,t) \in \Pi(\Gamma_\xi \cdot \mathcal{D})} \subset B_X$, $(x_t^*)_{t \in \text{MAX}(\Gamma_\xi \cdot \mathcal{D})} \subset K$, and $(C_s)_{s \in \Gamma_\xi \cdot \mathcal{D}} \subset \mathcal{K}$ such that for every $t \in \text{MAX}(\Gamma_\xi \cdot \mathcal{D})$,

$$\text{Re } x_t^* \left(\sum_{s \preceq t} \mathbb{P}_\xi(s) x_{(s,t)} \right) \geq \varepsilon$$

and for every $s \in \Gamma_\xi \cdot \mathcal{D}$ and every maximal extension t of s , $x_{(s,t)} \in C_s$. Fix $R > 0$ such that $K \subset RB_{X^*}$ and define the function

$$f : \Pi(\Gamma_\xi \cdot \mathcal{D}) \rightarrow [-R, R]$$

by $f(s, t) = \operatorname{Re} x_t^*(x_{(s,t)})$. For every $t \in \operatorname{MAX}(\Gamma_\xi \mathcal{D})$,

$$\sum_{s \preceq t} \mathbb{P}_\xi(s) f(s, t) = \operatorname{Re} x_t^* \left(\sum_{s \preceq t} \mathbb{P}_\xi(s) x_{(s,t)} \right) \geq \varepsilon.$$

By [5, Theorem 4.3], there exists an extended pruning $(\theta, e) : \Gamma_\xi \mathcal{D} \rightarrow \Gamma_\xi \mathcal{D}$ such that for every $(s, t) \in \Pi(\Gamma_\xi \mathcal{D})$, $\operatorname{Re} x_{e(t)}^*(x_{(\theta(s), e(t))}) = f(\theta(s), e(t)) \geq \varepsilon'$. Fix $\delta > 0$ such that $R\delta < \varepsilon' - \varepsilon_1$. We may apply Lemma 4.3 with this δ to the collection $(x_{(\theta(s), e(t))})_{(s,t) \in \Pi(\Gamma_\xi \mathcal{D})}$ and $(C_{\theta(s)})_{s \in \Gamma_\xi \mathcal{D}}$ to obtain another extended pruning $(\theta', e') : \mathcal{T}_{\omega^\xi} \mathcal{D} \rightarrow \Gamma_\xi \mathcal{D}$ and a normally weakly null collection $(x'_s)_{s \in \mathcal{T}_{\omega^\xi} \mathcal{D}} \subset B_X$ such that for every $s \in \mathcal{T}_{\omega^\xi} \mathcal{D}$ and every maximal extension t of s ,

$$\|x_s - x_{\theta \circ \theta'(s), e \circ e'(t)}\| < \delta.$$

Fix any maximal $t \in \mathcal{T}_{\omega^\xi} \mathcal{D}$ and note that $x_{e \circ e'(t)}^* \in K \subset RB_{X^*}$. For any $1 \leq i \leq |t|$,

$$\operatorname{Re} x_{e \circ e'(t)}^*(x'_{t|_i}) \geq \operatorname{Re} x_{e \circ e'(t)}^*(x_{\theta \circ \theta'(t|_i), e \circ e'(t)}) - R\|x'_{t|_i} - x_{\theta \circ \theta'(t|_i), e \circ e'(t)}\| \geq \varepsilon' - R\delta.$$

Since $\varepsilon' - R\delta > \varepsilon_1$, Remark 4.4 guarantees that $Sz(K, \varepsilon_1) > \omega^\xi$. □

Corollary 4.6. *Given an ordinal ξ and a w^* -compact set $K \subset X^*$, $Sz(K) > \omega^\xi$ if and only if there exists $\varepsilon > 0$ such that Player II has a winning strategy in the $\mathcal{E}_{K, \varepsilon}(\Gamma_\xi \mathcal{D}, \mathcal{K}, \mathbb{P}_\xi)$ game.*

4.2. Applications to essentially bounded trees in $L_p(X)$. We recall the following special case of the main theorem of [1].

Theorem 4.7 ([1]). *If X is a separable Banach space not containing ℓ_1 , then $Sz(X) > \omega$ if and only if there exists a B -tree B with $o(B) = \omega$ and a weakly null collection $(f_t)_{t \in B} \subset B_X$ such that for every $t \in B$ and $f \in \operatorname{co}(f_s : s \preceq t)$, $\|f\| \geq \varepsilon$.*

It is easy to see that $Sz(X) = 1$ if and only if X has finite dimension. It was shown in [8] that any asymptotically uniformly smooth Banach space has Szlenk index not exceeding ω , whence for any $1 < p < \infty$, $Sz(L_p) = \omega$. It is also easy to see that the Szlenk index is an isomorphic invariant, so that any Banach space isomorphic to L_p has Szlenk index ω .

Recall that for $1 < p < \infty$, $L_p(X)$ denotes the Banach space of (equivalence classes of) Bochner integrable functions $f : [0, 1] \rightarrow X$ such that $\int \|f\|^p < \infty$, where $[0, 1]$ is endowed with Lebesgue measure. We let $L_\infty(X)$ denote the X -valued strongly measurable functions which are essentially bounded. It is well known and easy to see that for any subspace Z of X , $L_p(X)/L_p(Z)$ is canonically isometrically isomorphic to $L_p(X/Z)$ by the operator Φ such that for each $\varpi \in [0, 1]$, $\Phi(f + L_p(Z))(\varpi) = f(\varpi) + Z$. Moreover, if $\dim X/Z < \infty$, $L_p(X/Z)$ is isomorphic to L_p and therefore has Szlenk index ω . This means that for any B -tree T with $o(T) \geq \omega$ and any weakly null collection $(\bar{f}_t)_{t \in T} \subset B_{L_p(X/Z)}$ and any $\delta > 0$, there exists $t \in T$ and a convex combination \bar{f} of $(\bar{f}_s : s \preceq t)$ such that $\|\bar{f}\| < \delta$. This means that if T is a B -tree with $o(T) \geq \omega$, $\dim X/Z < \infty$, $\delta > 0$, and if $(f_t)_{t \in T} \subset B_{L_p(X)}$ is a weakly null collection, there exists $t \in T$ and $f \in \operatorname{co}(f_s : s \preceq t)$ such that $\|f\|_{L_p(X)/L_p(Z)} < \delta$. Indeed,

we simply let $\bar{f}_t = f_t + L_p(Z)$ and use the previous fact, noting that $(\bar{f}_t)_{t \in T}$ is still weakly null and contained in $B_{L_p(X)/L_p(Z)}$ and using the isometric identification of $L_p(X)/L_p(Z)$ and $L_p(X/Z) \approx L_p$. Finally, if $f \in CB_{L_\infty(X)}$ and $\|f\|_{L_p(X)/L_p(Z)} < \delta$, then there exists a simple function $g \in 2CB_{L_\infty(Z)}$ such that $\|f - g\|_{L_p(X)} < \delta$. Indeed, we may first fix $h \in L_p(Z)$ such that $\|f - h\|_{L_p(X)} < \delta$ and, by density of simple functions in $L_p(Z)$, assume h is simple. Next, let $E = \{\varpi : \|h(\varpi)\| > 2C\}$. Note that there exists a subset N of E having measure zero such that for all $\varpi \in E \setminus N$,

$$\|f(\varpi)\| \leq C \leq \|h(\varpi)\| - \|f(\varpi)\| \leq \|h(\varpi) - f(\varpi)\|.$$

Thus we deduce that

$$\|f - 1_{E^c}h\|^p = \int_E \|f\|^p + \int_{E^c} \|f - h\|^p \leq \int_E \|f - h\|^p + \int_{E^c} \|f - h\|^p < \delta^p.$$

Thus $g = 1_{E^c}h$ is the simple function we seek.

Fix $1 < p < \infty$ and let q be the conjugate exponent to p . For a fixed $K \subset X^*$, let M denote the K -valued, measurable simple functions in $L_q(X^*)$. Recall that $L_q(X^*)$ is canonically isometrically included in $L_p(X)^*$ via the action $g(f) = \int g(\varpi)(f(\varpi))d\varpi$. Finally, let \mathcal{D}_0 denote the subspaces of $L_p(X)$ having finite codimension in $L_p(X)$.

Theorem 4.8. *With K, M as above, if $Sz(K) \leq \omega^\xi$, then for any B -tree S with $o(S) \geq \omega^{1+\xi}$, any weakly null collection of simple functions $(f_t)_{t \in S} \subset \frac{1}{2}B_{L_\infty(X)}$, and any $\varepsilon > 0$, there exist $t \in S$ and $f \in \text{co}(f_s : s \preceq t)$ such that $\sup_{h \in M} \text{Re} \int h f \leq \varepsilon$.*

Proof. Fix $R > 0$ such that $K \subset RB_{X^*}$ and note that $M \subset RB_{L_p(X)^*}$. By Proposition 3.1, the $\mathcal{E} = \mathcal{E}_{K, \varepsilon/2}(\Gamma_\xi, \mathcal{D}, \mathbb{P}_\xi)$ game on $\Gamma_\xi, \mathcal{D}, \mathbb{P}_\xi$ is determined. Since $Sz(K) \leq \omega^\xi$, Corollary 4.6 implies that Player II cannot have a winning strategy, and therefore Player I has a winning strategy. Fix a winning strategy φ for Player I. Define $m : \Gamma_\xi, \mathcal{D} \rightarrow [0, \omega^\xi]$ by letting $m(t) = \max\{\gamma < \omega^\xi : t \in (\Gamma_\xi, \mathcal{D})^\gamma\}$.

We next define several sequences recursively. Let $\varphi(\emptyset) = (\zeta_1, Z_1)$. Let $\gamma_1 = m(\zeta_1, Z_1)$. Note that since $\gamma_1 < \omega^\xi$ and $o(S) \geq \omega^{1+\xi}$, $o(S^{\omega\gamma_1}) \geq \omega$ and $(f_t)_{t \in S^{\omega\gamma_1}} \subset \frac{1}{2}B_{L_\infty(X)}$ is normally weakly null. By the remarks in the paragraphs preceding the statement of the theorem, there exist $s_1 \in S^{\omega\gamma_1}$, a convex combination f_1 of $(f_t : t \preceq s_1)$, and a simple function $g_1 \in B_{L_\infty(Z_1)}$ such that $\|f_1 - g_1\|_{L_p(X)} < \varepsilon/2R$. By redefining g_1 on a set of measure zero, we may assume $\text{range}(g_1) \subset B_{Z_1}$ is finite. Let $C_1 = \text{range}(g_1) \subset B_X$.

Next, suppose that for each $1 \leq i \leq n$, $\zeta_i, Z_i, C_i, s_i, \gamma_i, f_i, g_i$ have been defined to have the following properties:

- (i) $\varphi((\zeta_j, Z_j, C_j)_{j=1}^{i-1}) = (\zeta_i, Z_i)$,
- (ii) $\|f_i - g_i\|_{L_p(X)} < \varepsilon/2R$,
- (iii) $C_i = \text{range}(g_i) \subset B_{Z_i}$ is finite,
- (iv) $\gamma_i = m((\zeta_j, Z_j)_{j=1}^i)$,
- (v) $f_i \in \text{co}(f_s : s_{i-1} \prec s \preceq s_i)$, (where $s_0 = \emptyset$).

If $(\zeta_i, Z_i)_{i=1}^n$ is maximal in $\Gamma_\xi \mathcal{D}$, we have completed the recursive construction. Suppose that $(\zeta_i, Z_i)_{i=1}^n$ is not maximal in $\Gamma_\xi \mathcal{D}$. Let $\varphi((\zeta_i, Z_i, C_i)_{i=1}^n) = (\zeta_{n+1}, Z_{n+1})$. Let $\gamma_{n+1} = m((\zeta_i, Z_i)_{i=1}^{n+1})$. Let U denote those non-empty sequences s such that $s_{\widehat{n}} s \in S^{\omega_{\gamma_{n+1}}}$. Applying the remarks in the paragraphs preceding the proof to the collection $(f_{s_{\widehat{n}} s})_{s \in U}$, we deduce the existence of $s_{n+1} \in S^{\omega_{\gamma_{n+1}}}$, $f_{n+1} \in \text{co}(f_s : s_n \prec s \preceq s_{n+1})$, and $g_{n+1} \in B_{L_\infty(Z_{n+1})}$ such that $\|f_{n+1} - g_{n+1}\|_{L_p(X)} < \varepsilon/2R$. Here we have used that since $s_n \in S^{\omega_{\gamma_n}}$ and $\gamma_{n+1} < \gamma_n$, $o(U) \geq \omega$. By redefining g_{n+1} on a set of measure zero, we may assume $\text{range}(g_{n+1}) \subset B_{Z_{n+1}}$ is finite. Let $C_{n+1} = \text{range}(g_{n+1})$.

Since $\Gamma_\xi \mathcal{D}$ is well-founded, this process must eventually terminate. Assume that the process terminates with the sequence $(\zeta_i, Z_i)_{i=1}^n \in \text{MAX}(\Gamma_\xi \mathcal{D})$, the sequences s_i , and the functions f_i, g_i . By our choices, $(\zeta_i, Z_i, C_i)_{i=1}^n$ is φ -admissible, and therefore not a member of $\mathcal{E}_{K, \varepsilon/2}(\Gamma_\xi \mathcal{D}, \mathcal{K}, \mathbb{P}_\xi)$. This means that for any $(x_i)_{i=1}^n \in \prod_{i=1}^n (B_X \cap Z_i \cap C_i)$ and for all $x^* \in K$, $\text{Re } x^* \left(\sum_{i=1}^n \mathbb{P}_\xi(t|i)x_i \right) < \varepsilon/2$. But for any $\varpi \in [0, 1]$, $(g_i(\varpi))_{i=1}^n \in \prod_{i=1}^n (B_X \cap Z_i \cap C_i)$, whence for any $x^* \in K$, $\text{Re } x^* \left(\sum_{i=1}^n \mathbb{P}_\xi(t|i)g_i(\varpi) \right) < \varepsilon/2$. Then with $g = \sum_{i=1}^n \mathbb{P}_\xi(t|i)g_i$ and $h \in M$, $\text{Re} \int hg \leq \varepsilon/2$, whence

$$\sup_{h \in M} \text{Re} \int hg \leq \varepsilon/2.$$

Let $f = \sum_{i=1}^n \mathbb{P}_\xi(t|i)f_i \in \text{co}(f_s : s \preceq s_n)$ and note that

$$\|f - g\|_{L_p(X)} \leq \sum_{i=1}^n \mathbb{P}_\xi(t|i)\|f_i - g_i\|_{L_p(X)} < \varepsilon/2R.$$

Since $M \subset RB_{L_p(X)^*}$, it follows that

$$\sup_{h \in M} \text{Re} \int hf \leq \sup_{h \in M} \text{Re} \int hg + R\|f - g\|_{L_p(X)} \leq \varepsilon.$$

□

5. THE w^* -DENTABILITY INDEX AND A RESULT OF LANCIEN

In this section, we again fix $1 < p < \infty$ and let q be the conjugate exponent to p . Let \mathcal{W} be a w^* -neighborhood basis at 0 in $L_p(X)^*$. The following was shown in [4] in the case that L is w^* -compact. However, the proof given there does not depend upon the w^* -compactness of L . For the remainder of the section, $K \subset X^*$ will be a fixed w^* -compact, non-empty set and M will denote the subset of $L_q(X^*) \subset L_p(X)^*$ consisting of all K -valued, measurable simple functions.

Proposition 5.1. *For an ordinal ξ , if $h \in s_{2^\xi}^\xi(L)$, there exists a collection $(h_t)_{t \in (\mathcal{T}_\xi \cup \{\emptyset\}) \cdot \mathcal{W}} \subset L$ such that $h_\emptyset = h$ and for every $t \in \mathcal{T}_\xi \cdot \mathcal{W}$, if $t = (\zeta_i, V_i)_{i=1}^n$, $\|h_t - h_{t-}\|_{L_p(X)^*} > \varepsilon$ and $h_t - h_{t-} \in V_n$.*

A collection $(h_t)_{t \in (\mathcal{T}_\xi \cup \{\emptyset\}, \mathcal{W})}$ satisfying the condition that for any $t \in \mathcal{T}_\xi \mathcal{W}$, if $t = (\zeta_i, V_i)_{i=1}^n$, then $h_t - h_{t-} \in V_n$ will be called *normally w^* -closed*. A collection such that for any $t \in \mathcal{T}_\xi \mathcal{W}$, $\|h_t - h_{t-}\| > \varepsilon$ will be called *ε -separated*.

Although it was not stated in this way, the following theorem was shown in [14]. Since the statement of this theorem differs significantly from the statement in [14], we will sketch the statement here for completeness.

Theorem 5.2. [14, Lemma1] *Suppose that K is convex. If $n \in \mathbb{N}$ and $x_1^*, \dots, x_n^* \in d_{2\varepsilon}^\xi(K)$, then $\sum_{i=1}^n x_i^* 1_{[\frac{i-1}{n}, \frac{i}{n})} \in s_\varepsilon^\xi(M)$.*

Sketch. Given $K_0 \subset X^*$ and $L_0 \subset L_p(X)^*$, let us say the pair (K_0, L_0) is *nice* provided that K_0 is w^* -compact, convex, and symmetric, and for any $n \in \mathbb{N}$ and $x_1^*, \dots, x_n^* \in K_0$, $\sum_{i=1}^n x_i^* 1_{[\frac{i-1}{n}, \frac{i}{n})} \in L_0$. Of course, if (K_0, L_0) is nice and $K_0 \neq \emptyset$, then $L_0 \neq \emptyset$. We claim that if (K_0, L_0) is nice, then for any $\varepsilon > 0$, the pair $(d_{2\varepsilon}(K_0), s_\varepsilon(L_0))$ is nice. An easy induction then yields that for any ordinal ξ , the pair $(d_{2\varepsilon}^\xi(K_0), s_\varepsilon^\xi(L_0))$ is nice, whence $Dz(K_0, 2\varepsilon) \leq Sz(L_0, \varepsilon)$. We obtain Theorem 5.2 by noting that (K, M) is nice.

We prove the claim that $(d_{2\varepsilon}(K_0), s_\varepsilon(L_0))$ is nice, assuming (K_0, L_0) is nice. Of course, $d_{2\varepsilon}(K_0)$ is w^* -compact, convex, and symmetric. Fix $n \in \mathbb{N}$ and $x_1^*, \dots, x_n^* \in d_{2\varepsilon}(K_0)$. Let $f = \sum_{i=1}^n x_i^* 1_{[\frac{i-1}{n}, \frac{i}{n})} \in L_0$. Let V be a w^* -open neighborhood of f . It follows by the Hahn-Banach theorem that for each $1 \leq i \leq n$, x_i^* lies in the w^* -closed, convex hull of $K_0 \setminus (x_i^* + \varepsilon B_{X^*})$. Then there exist $k \in \mathbb{N}$ and $(x_{ij}^*)_{i=1, j=1}^{n, k} \subset K$ such that

$$g = \sum_{i=1}^n \left(k^{-1} \sum_{j=1}^k x_{ij}^* \right) 1_{[\frac{i-1}{n}, \frac{i}{n})} \in V.$$

For each $l \in \mathbb{N}$, let

$$\psi_l = \sum_{i=1}^n \sum_{j=1}^k \sum_{m=1}^l x_{ij}^* 1_{I_{ijm}},$$

where

$$I_{ijm} = \left[\frac{i-1}{n} + \frac{m-1}{nl} + \frac{j-1}{nlk}, \frac{i-1}{n} + \frac{m-1}{nl} + \frac{j}{nlk} \right).$$

Note that $\psi_l \xrightarrow{w^*} g$, whence $\psi_l \in V$ for sufficiently large $l \in \mathbb{N}$. Since (K_0, L_0) is nice, $\psi_l \in L_0$ for all $l \in \mathbb{N}$. Also, for any $\varpi \in [0, 1]$, $\|f(\varpi) - \psi_l(\varpi)\| > \varepsilon$, whence $\|f - \psi_l\|_{L_p(X)^*} > \varepsilon$. This shows that $f \in s_\varepsilon(L_0)$. □

We remark that if $h \in L_q(X^*)$ is a simple function such that $\|h\|_{L_q(X^*)} > \varepsilon > 0$ and $\|h\|_{L_\infty(X^*)} \leq C$, there exists a simple function $f \in B_{L_p(X)}$ with $\|f\|_{L_\infty(X)} \leq C^{q-1}/\varepsilon^{q-1}$ and $\int hf > \varepsilon$. Indeed, write $h = \sum_{i=1}^n x_i^* 1_{F_i}$ with F_i pairwise disjoint and measurable. Fix $0 < \rho < 1$ such that $\rho \|h\|_{L_q(X^*)} > \varepsilon$. For each $1 \leq i \leq n$, fix $x_i \in S_X$ such that $x_i^*(x_i) > \rho \|x_i^*\|$. Then $f = \|h\|_{L_q(X^*)}^{1-q} \sum_{i=1}^n \|x_i^*\|^{q-1} x_i 1_{F_i}$ has the indicated properties by familiar computations.

Lemma 5.3. *Suppose that $R \geq 1$ is such that $K \subset RB_{X^*}$. Assume $(h_t)_{t \in (\mathcal{T}_\xi \cup \{\emptyset\}) \cdot \mathcal{W}} \subset M$ is normally w^* -closed and ε -separated for some $\varepsilon \in (0, 1)$. Then there exist a function $\theta : \mathcal{T}_\xi \cdot \mathcal{D} \rightarrow \mathcal{T}_\xi \cdot \mathcal{W}$ and a weakly null collection $(f_t)_{t \in \mathcal{T}_\xi \cdot \mathcal{N}} \subset \frac{1}{2}B_{L_\infty(X)}$ such that for any $\emptyset \prec s \preceq t$,*

$$\operatorname{Re} \int h_{\theta(t)} f_s \geq \frac{\varepsilon^q}{3 \cdot 2^q R^{q-1}}.$$

Here, \mathcal{N} denotes the directed set of convex, weakly open neighborhoods of 0 in $L_p(X)$.

Proof. We will need the following claim.

Claim 5.4. *If $(h_t)_{t \in (\mathcal{T}_\xi \cup \{\emptyset\}) \cdot \mathcal{W}} \subset M$ is normally w^* -closed and ε -separated, then for any sequence $(\varepsilon_n)_{n=0}^\infty$ of positive numbers, there exists a monotone, length-preserving function $\theta : \mathcal{T}_\xi \cdot \mathcal{D}_0 \rightarrow \mathcal{T}_\xi \cdot \mathcal{W}$ and a collection $(g_t)_{t \in \mathcal{T}_\xi \cdot \mathcal{N}} \subset \frac{2^{q-1} R^{q-1}}{\varepsilon^{q-1}} B_{L_\infty(X)}$ such that for every $s \in \mathcal{T}_\xi \cdot \mathcal{N}$, $\operatorname{Re} \int h_{\theta(t)} g_s > \varepsilon/2 - \varepsilon_0$ and such that for any $\emptyset \prec s \prec t$, $|\int (h_{\theta(t)} - h_{\theta(t-)}) g_s| < \varepsilon_{|t|}$, and if $s = (\zeta_i, U_i)_{i=1}^n$, $g_s \in U_n$.*

We first assume the claim and finish the proof. We apply the claim with some sequence $(\varepsilon_n)_{n=0}^\infty$ such that $\varepsilon/2 - \sum_{n=0}^\infty \varepsilon_n > \varepsilon/3$. Fix any $t \in \mathcal{T}_\xi \cdot \mathcal{D}_0$ and let $\emptyset \prec s \preceq t$. Then

$$\begin{aligned} \operatorname{Re} \int h_{\theta(t)} g_s &\geq \operatorname{Re} \int h_{\theta(s)} g_s - \sum_{s \prec u \preceq t} |\int (h_{\theta(u)} - h_{\theta(u-)}) g_s| > \varepsilon/2 - \varepsilon_0 - \sum_{n=|s|+1}^{|t|} \varepsilon_n \\ &> \varepsilon/2 - \sum_{n=0}^\infty \varepsilon_n > \varepsilon/3. \end{aligned}$$

From this it follows that for any convex combination g of $(g_s : s \preceq t)$, $\operatorname{Re} \int h_{\theta(t)} g > \varepsilon/3$. Letting $f_t = \frac{\varepsilon^{q-1}}{2^q R^{q-1}} g_t$ gives the desired collection. Since $\varepsilon^{q-1}/2^q R^{q-1} \in (0, 1)$, if $s = (\zeta_i, U_i)_{i=1}^n$, $f_s \in U_n$, since U_n is a convex weak neighborhood of 0 and $g_s \in U_n$. This condition guarantees that the collection $(f_t)_{t \in \mathcal{T}_\xi \cdot \mathcal{N}}$ is weakly null.

We return to the proof of the claim. We define g_s and $\theta(s)$ by induction on $|s|$, and we define θ to be monotone, length-preserving, and so that for any $s = (\zeta_i, U_i)_{i=1}^n$, $\theta(s) = (\zeta_i, V_i)_{i=1}^n$ for some $V_1, \dots, V_n \in \mathcal{W}$. Note that these properties together imply that if $\theta(s) = (\zeta_i, V_i)_{i=1}^n$ and $1 \leq m \leq n$, $\theta(s|_m) = (\zeta_i, V_i)_{i=1}^m$.

Suppose $(h_t)_{t \in (\mathcal{T}_\xi \cup \{\emptyset\}) \cdot \mathcal{W}} \subset M$ is as in the claim. Fix some $(\zeta, U) \in \mathcal{T}_\xi \cdot \mathcal{N}$. For every $V \in \mathcal{W}$, $\|h_{(\zeta, V)} - h_\emptyset\|_{L_q(X^*)} > \varepsilon$ and $h_{(\zeta, V)} - h_\emptyset$ is a simple function with $\|h_{(\zeta, V)} - h_\emptyset\|_{L_\infty(X^*)} \leq 2R$. By the remarks preceding the lemma, there exists a simple function $j_V \in B_{L_p(X)} \cap \frac{2^{q-1} R^{q-1}}{\varepsilon^{q-1}}$ such that $\operatorname{Re} \int (h_{(\zeta, V)} - h_\emptyset) j_V > \varepsilon$. By [7, Lemma 3.3], for each $U \in \mathcal{N}$, there exist $V_1^U, V_2^U \in \mathcal{W}$ such that

$$\operatorname{Re} \int h_{(\zeta, V_2^U)} \left(\frac{j_{V_2^U} - j_{V_1^U}}{2} \right) > \varepsilon/2 - \varepsilon_0$$

and

$$\frac{j_{V_2^U} - j_{V_1^U}}{2} \in U.$$

We let $g_{(\zeta, U)} = \frac{j_{V_2^U} - j_{V_1^U}}{2}$ and $\theta((\zeta, U)) = (\zeta, V_2^U)$.

Now suppose that for some $s = s_1^\wedge(\eta, W) \in \mathcal{T}_\xi \mathcal{N}$ with $s_1 \neq \emptyset$, and for every $\emptyset \prec u \preceq s_1$, g_u and $\theta(u)$ have been defined to have the indicated properties. Let $t = \theta(s_1)$. For every $V \in \mathcal{W}$, $\|h_{t^\wedge(\eta, V)} - h_t\| > \varepsilon$, $\|h_{t^\wedge(\eta, V)} - h_t\|_{L_\infty(X^*)} \leq 2R$, and the function $h_{t^\wedge(\eta, V)} - h_t$ is simple, whence there exists a simple function $i_V \in B_{L_p(X)}$ with $\|i_V\|_{L_\infty(X)} \leq 2^{q-1}R^{q-1}/\varepsilon^{q-1}$ such that $\operatorname{Re} \int (h_{t^\wedge(\eta, V)} - h_t)i_V > \varepsilon$. Again using [7, Lemma 3.3], there exist $V_1^W, V_2^W \in \mathcal{W}$ such that

$$\operatorname{Re} \int h_{t^\wedge(\eta, V_2^W)} \left(\frac{i_{V_2^W} - i_{V_1^W}}{2} \right) > \varepsilon/2 - \varepsilon_0,$$

$$\frac{i_{V_2^W} - i_{V_1^W}}{2} \in W,$$

and

$$V_2^W \subset \{h \in L_p(X)^* : (\forall \emptyset \prec u \preceq s_1)(|h(g_u)| < \varepsilon_{|s|})\}.$$

We let $g_s = \frac{i_{V_2^W} - i_{V_1^W}}{2}$ and $\theta(s) = t^\wedge(\eta, V_2^W)$. This finishes the construction, and the conclusions of the claim are easily verified. \square

Corollary 5.5. *If K is convex and $Dz(K) > \omega^{1+\xi}$, then there exists a constant $\varepsilon' > 0$ and a weakly null collection $(f_t)_{t \in \mathcal{T}_{\omega^{1+\xi}} \mathcal{N}} \subset \frac{1}{2}B_{L_\infty(X)}$ such that for every $t \in \mathcal{T}_{\omega^{1+\xi}} \mathcal{D}_0$ and every convex combination f of $(f_s : s \preceq t)$,*

$$\sup_{h \in M} \operatorname{Re} \int h f \geq \varepsilon'.$$

6. PROOF OF THE MAIN THEOREM

Proof of Theorem 1. Let K denote the w^* -closed, convex, symmetrized hull of K_0 . By [5, Theorem 1.5], $Sz(K) \leq \omega^\xi$. If $Dz(K) > \omega^{1+\xi}$, there exists a constant $\varepsilon' > 0$ and a normally weakly null collection $(f_t)_{t \in \mathcal{T}_{\omega^{1+\xi}} \mathcal{N}} \subset \frac{1}{2}B_{L_\infty(X)}$ as in the conclusion of Corollary 5.5. By Theorem 4.8, the existence of such a collection implies that $Sz(K) > \omega^\xi$. It follows that $Dz(K) \leq \omega^{1+\xi}$, whence $Dz(K_0) \leq Dz(K) \leq \omega^{1+\xi}$. This gives (i).

For (ii), note that if K is convex, $Sz(K) = \omega^\xi$ for some ordinal ξ , or $Sz(K) = \infty$ if K is not w^* -fragmentable. In the first case, by (i), we deduce that $Dz(K) \leq \omega^{1+\xi} = \omega Sz(K)$. If K is not w^* -fragmentable, it is not w^* -dentable, and $Dz(K) = \infty = \omega\omega = \omega Sz(K)$ by convention. If $Sz(K) \geq \omega^\omega$, then either $Sz(K) = Dz(K) = \infty$ or $Sz(K) = \omega^\xi$ and $Dz(K) \leq \omega^{1+\xi}$ for an ordinal $\xi \geq \omega$. But since $\xi \geq \omega$, $1 + \xi = \xi$, and $Dz(K) \leq \omega^{1+\xi} = \omega^\xi = Sz(K)$. \square

As we have already mentioned, for every $n \in \mathbb{N} \cup \{0\}$, there exist a pair of Banach spaces X_n, Y_n such that $Sz(X_n) = Dz(X_n) = \omega^n$ and $Dz(Y_n) = \omega Sz(Y_n) = \omega^{n+1}$, so that Theorem 1 is sharp.

If $A : X \rightarrow Y$ is an operator, for any $1 < p < \infty$, A induces an operator $A_p : L_p(X) \rightarrow L_p(Y)$ such that for any $\varpi \in [0, 1]$, $(A_p f)(\varpi) = A(f(\varpi))$. Since $(A^* B_{Y^*}, (A_p)^* B_{L_p(Y)^*})$ is

nice, Theorem 5.2 yields that $Dz(A) \leq Sz(A_p)$. Thus a positive solution to the next question implies Theorem 1.

Question 6.1. *For any operator $A : X \rightarrow Y$ and $1 < p < \infty$, is it true that $Sz(A_p) \leq \omega Sz(A)$?*

By [11], Question 6.1 has a positive answer when A is an identity operator and $Sz(A)$ is countable. It is possible to deduce using arguments similar to those in [11] that if $Sz(A)$ is countable, Question 6.1 has a positive answer.

A positive solution to the following question would imply a positive solution to Question 6.1.

Question 6.2. *For any operator $A : X \rightarrow Y$ and $1 < p < \infty$, is it true that $Dz(A) = Sz(A_p)$?*

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